Charged black holes: Wave equations for gravitational and electromagnetic perturbations

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ABSTRACT

A pair of wave equations for the electromagnetic and gravitational perturbations of the charged Kerr black hole are derived. The perturbed Einstein-Maxwell equations in a new gauge are employed in the derivation. The wave equations refer to the perturbed Maxwell spinor Φ_0 and to the shear σ of a principal null direction of the Weyl curvature. The whole construction rests on the tripod of three distinct derivatives of the first curvature κ of a principal null direction.

1. Introduction

One of the most fundamental issues of theoretical astrophysics concerns the relativistic sources of radiation involving black holes. The basic properties of black holes (their geometry, unicity and thermodynamical properties) have been discovered in the middle of the twentieth century. Early investigations by Petterson Petterson (1975), Chitre and Vishveshwara Chitre (1975) indicate that the minimum energy configuration of a black hole in an external electromagnetic field is attained when the hole has an electric charge.

The recently discovered high-luminosity sources of X-ray radiation (Bagaroff 2001; Vazquez 2002) and the need for templates of gravitational radiation wave forms have amplified interest (Hughes 2000) in models of black-hole perturbations (Wagoner 2001). It has been suggested (Ruffini 2001; Punsly 2001) that the observed features of gamma-ray bursters (GRB) can be modelled by the presence of an electrically charged black hole. There is now an abundant literature of this subject. Preparata and his collaborators Preparata (1998) present a model of GRB 971214.

The relativistic description of radiative processes follows two distinct threads: the post-Newtonian expansion and black-hole perturbation theory. Despite the recent advances in the post-Newtonian treatment of gravitational radiation processes, it remains a laborious apparatus to work with. In situations involving a black hole, it would seem unreasonable to perturb the flat Minkowski background when the black hole metric is known to full precision.

The first investigation of black hole perturbations has been undertaken by Chrzanowski and Misner (1974), Detweiler (1977) and Chandrasekhar (1983). In the focus of their descriptions is the decoupled equation (Teukolsky 1973) for the perturbed curvature quantity Ψ_0. The ordinary differential equations resulting from the separation of this equation are of second order, with a rugged singularity structure.

Among these pioneering studies, Chandrasekhar's own, reviewed in Sec. 9 of his monograph (Chandrasekhar 1983), stands out as most detailed. In a *tour de force* of several hundred computation pages, Chandrasekhar manages to solve his seventy-six perturbation equations. He adds, however, the conjecture that perhaps at a later time the complexity of the problem will be unravelled by deeper insight.

Chandrasekhar uses a Newman and Penrose (1962) (NP) approach both to the unperturbed Kerr metric and to the perturbed space-time. The null tetrad vectors of the latter are linear combinations of those for the black-hole background. The combination coefficients plus the Weyl tensor perturbations are fifty real unknown functions.

The study of electromagnetic black hole perturbations has begun with the work of ZerilliZerilli (1974) who considered the nonrotating system. Some years later, the problem of charged Kerr black hole perturbations has been taken up by Fackerell Fackerell (1982) and Crossman Crossman (1976).

The purpose of the present work is precisely to help unravel the mystery of black-hole perturbations.

In treatments of black-hole perturbations, it is a frequent practice to seek a suitable gauge fixing. The commonly applied gauge uses the Kinnersley (1969) tetrad (Chitre 1976). Another possible choice, promoted by Chrzanowski (1975), is the incoming (or outgoing) radiation gauge for the normal modes. This choice does not uniquely fix the coordinate gauge, however. In fact, the present work has been launched with the intent to take a second look at the remaining gauge freedom. We are lead to use, however a tetrad gauge related to the timelike Killing vector K.

In Perjés (2002a), a simple pilot computation has revealed for stationary perturbations of the charged Kerr black hole that it is possible to derive a pair of wave equations for the electromagnetic and for the gravitational fields in a gauge related to the timelike Killing vector $K = \partial/\partial t$. We now show that this result can be extended to arbitrary perturbations when using the Newman et al. (1965) tetrad. To our delight, the two wave equations governing the Maxwell field function Φ_0 and the shear σ of a principal null direction of the Weyl tensor survive the generalization to arbitrary perturbations! A brief account of this latter result has been published in Perjés (2002b). The present paper discusses the full details.

In Section 2, we collect the relations describing the Kerr-Newman solution of the coupled Einstein-Maxwell equations. In section 3, we consider the problem of choosing the tetrad for the perturbed space-time, and derive a triple of expressions for three derivatives of the first curvature κ of a principal null direction. The whole description of black-hole perturbations rests on this tripod. In Section 4, the integrability conditions of these derivatives will provide us with the key wave equations. The boundary conditions on the horizon and at null infinity are studied in Section 5.

2. The unperturbed variables

The current literature of black hole physics predominantly utilizes the Boyer-Lindquist coordinates $\left(\hat{t}, r, \vartheta, \hat{\varphi}\right)$ which are related to the Kerr-Newman coordinates $(t, r, \vartheta, \varphi)$ by the transformation

$$\begin{aligned} d\hat{t} &= -dt + \tfrac{r^2+a^2}{\boldsymbol{\Delta}} dr \\ d\hat{\varphi} &= -d\varphi + \tfrac{a}{\boldsymbol{\Delta}} dr \end{aligned} \tag{1}$$

where

$$\boldsymbol{\Delta} = r^2 - 2\mathrm{m}r + a^2 + \mathrm{e}^2 \tag{2}$$

is the horizon function. The effect of the transformation is a simultaneous reflection of the time and lattitude directions, and a shift of the origin of the t and φ coordinates along the open t trajectories and circular φ trajectories, respectively, by an amount dependent on the radial coordinate r. Although the Boyer-Lindquist coordinates have the salient feature that the two double principal null directions of the curvature are represented symmetricallyMisner (1973), they cover only the part of the original $(t, r, \vartheta, \varphi)$ coordinate domain extending between null infinity and the horizon. (However, this symmetry is restored in the original coordinates by using a twin transformation to Eqs. (1) where the directions on the trajectories are not reversed).

In the original coordinates the charged Kerr metric is (Newman et al. 1965)

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\mathrm{m}r - \mathrm{e}^2}{\zeta\overline{\zeta}}\right)\left(dt - a\sin^2\vartheta d\varphi\right)^2 \\ &+ 2\left(dt - a\sin^2\vartheta d\varphi\right)\left(dr + a\sin^2\vartheta d\varphi\right) - \zeta\overline{\zeta}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \end{aligned} \tag{3}$$

with m the mass, e the electric charge, ma the angular momentum and

$$\zeta = r - ia\cos\vartheta.$$

An overbar denotes complex conjugation. The corresponding null tetrad is (Newman et al. 1965)

$$
\begin{aligned}
D &\equiv \ell^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r} \\
\Delta &\equiv n^a \frac{\partial}{\partial x^a} = \frac{1}{2}\left[\frac{\left(\zeta+\overline{\zeta}\right)\mathrm{m}-\mathrm{e}^2}{\zeta\overline{\zeta}} - 1\right]\frac{\partial}{\partial r} + \frac{\partial}{\partial t} \\
\delta &\equiv m^a \frac{\partial}{\partial x^a} = \frac{1}{2^{1/2}\overline{\zeta}}\left[\frac{\partial}{\partial\vartheta} + \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi} - ia\sin\vartheta\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial t}\right)\right] \\
\overline{\delta} &\equiv \overline{m}^a \frac{\partial}{\partial x^a}.
\end{aligned}
\tag{4}
$$

The real null vector ℓ of this tetrad points in one of the double principal directions of the space-time, and the vector n lies in the intersection of the two-plane of the vector ℓ and the vector K with the null cone. (Note that the literature of black hole perturbations is biased towards the use of the Kinnersley tetrad Misner (1973) as a companion of the Boyer-Lindquist coordinates, which we do not use in this paper). In the NP notation, the following spin coefficients have nonzero values

$$
\begin{aligned}
\rho &= -\frac{1}{\zeta} & \gamma &= \frac{\mathrm{m}\overline{\zeta}-\mathrm{e}^2}{2\zeta^2\overline{\zeta}} \\
\alpha &= -\frac{1}{2^{3/2}\zeta}\frac{\cos\vartheta}{\sin\vartheta} & \beta &= \frac{1}{2^{3/2}\overline{\zeta}}\frac{\cos\vartheta}{\sin\vartheta} \\
\mu &= \frac{\mathrm{m}(\zeta+\overline{\zeta})-\mathrm{e}^2}{2\zeta^2\overline{\zeta}} - \frac{1}{2\overline{\zeta}} & \nu &= ia\sin\vartheta\frac{\mathrm{m}\overline{\zeta}-\mathrm{e}^2}{2^{1/2}\zeta^3\overline{\zeta}}.
\end{aligned}
\tag{5}
$$

The nonvanishing components of the Weyl spinor are

$$
\begin{aligned}
\Psi_2 &= \frac{\mathrm{e}^2-\mathrm{m}\overline{\zeta}}{\zeta^3\overline{\zeta}} \\
\Psi_3 &= -3ia\sin\vartheta\frac{\mathrm{m}\overline{\zeta}-\mathrm{e}^2}{2^{1/2}\zeta^4\overline{\zeta}} \\
\Psi_4 &= 3a^2\sin^2\vartheta\frac{\mathrm{m}\overline{\zeta}-\mathrm{e}^2}{\zeta^5\overline{\zeta}}
\end{aligned}
\tag{6}
$$

and the nonzero Maxwell spinor components have the form

$$
\begin{aligned}
\Phi_1 &\equiv \frac{1}{2}F_{ab}\left(\ell^a n^b + \overline{m}^a m^b\right) = \frac{\mathrm{e}}{2^{1/2}\zeta^2} \\
\Phi_2 &\equiv F_{ab}\overline{m}^a n^b = \frac{i\mathrm{e}a\sin\vartheta}{\zeta^3}.
\end{aligned}
\tag{7}
$$

In the following sections, we develop a perturbative approach to the charged Kerr black hole. We shall focus on the small quantities in the perturbed space-time. It is, therefore, important to realize that the NP quantities $\sigma, \kappa, \epsilon, \lambda, \pi, \tau, \bar{\alpha}+\beta, \Psi_0, \Psi_1$ and Φ_0 all vanish in the present gauge.

3. Choice of gauge

In this section, we introduce a new gauge for the coupled electromagnetic and gravitational perturbations of the charged Kerr black hole in which the wave equations decouple, as found in the presence of a Killing symmetry (Perjés 2002a).

The vector ℓ is chosen along a principal null direction of the perturbed space-time:

$$\Psi_0 = 0. \tag{8}$$

Unlike in the case of the principal null spinor of the exact Weyl tensor, this condition does not determine uniquely the spinor o^A. Infinitesimal spinor rotations of the form

$$o^A \to o^A + b\iota^A, \qquad \iota^A \to \iota^A, \tag{9}$$

where b is an arbitrary first-order multiplying function, are allowed. The tetrad and spin coefficient transformations induced by these rotations are given in Eqs. (II) and (343) of Ref. Chandrasekhar (1983), respectively.

The gauge condition (8) is preserved by the dyad transformation (9). In this gauge, the electromagnetic Stewart-Walker identity (Stewart 1974)

$$\begin{aligned}&[(D-\epsilon+\bar{\epsilon}-2\rho-\bar{\rho})(\Delta+\mu-2\gamma)-(\delta-\beta-\bar{\alpha}-2\tau+\bar{\pi})(\bar{\delta}+\pi-2\alpha)+\sigma\lambda-\kappa\nu]\Phi_0 = \\ &-2[\kappa\,(\Delta-3\gamma-\bar{\gamma}+\bar{\mu})-\sigma\left(\bar{\delta}-3\alpha+\bar{\beta}-\bar{\tau}\right)+\Delta\kappa-\bar{\delta}\sigma+2\Psi_1]\Phi_1+\Psi_0\Phi_2.\end{aligned} \tag{10}$$

takes a simpler form since the last term on the right vanishes. The function b is then utilised, following Crossman's idea (Crossman 1976), to make the whole right-hand side vanish. The latter condition is employed to express the derivative

$$\Delta\kappa = \left(\bar{\delta} - 3\alpha + \bar{\beta} - \bar{\tau} + \frac{\bar{\delta}\Phi_1}{\Phi_1}\right)\sigma + \left(3\gamma + \bar{\gamma} - \bar{\mu} - \frac{\Delta\Phi_1}{\Phi_1}\right)\kappa - 2\Psi_1. \tag{11}$$

In what follows, we use a linear approximation, that is to say, we neglect products and higher powers of quantities which vanish for the Kerr-Newman space-time. The operators in the parentheses in Eq. (11) act on the first-order functions σ and κ, hence we keep their

form for the charged Kerr metric. For example, $\Phi_1 = \mathrm{e}/\left(2^{1/2}\zeta^2\right)$, where the numerator (the electric charge e) exactly cancels in the bracketed terms containing the field Φ_1. Thus we may keep our gauge condition (11) in the limiting case of an uncharged black hole, *i.e.*, when $\mathrm{e} = 0$, even though nothing remains of the original motivation for adopting it. In Chandrasekhar's terms (Chandrasekhar 1983), the gauge (11) becomes a phantom gauge.

Throughout, we express the derivatives $\delta\rho, D\rho, \Delta\rho, D\bar{\alpha}, D\tau, D\mu$ and $D\gamma$ from the NP equations (4.2k), (4.2a), (4.2q), (4.2d), (4.2c), (4.2h) and (4.2f), respectively; $D\Phi_2$ and $\delta\Phi_2$ from Maxwell's equations (A1) and $D\Psi_2$ and $\delta\Psi_2$ from the Bianchi identities (A3) (Newman and Penrose 1962).

A relation for $D\kappa$ can be obtained from the NP commutator (Newman and Penrose 1962)

$$\delta D - D\delta = \left(\bar{\alpha} + \beta - \bar{\pi}\right) D + \kappa\Delta - \sigma\bar{\delta} - \left(\bar{\rho} + \epsilon - \bar{\epsilon}\right)\delta \tag{12}$$

acting on Ψ_1, after expressing both $D\Psi_1$ and $\delta\Psi_1$ from the Bianchi identities. This has the form

$$\begin{aligned} D\kappa = {} & 2\frac{\bar{\Phi}_1}{\bar{\Phi}_2} D\sigma - \kappa\rho - \frac{1}{2\Phi_1}\ \left(D - 4\rho\right) D\Phi_0 \\ & + \frac{\bar{\Phi}_1}{2\bar{\Phi}_2\Phi_1}\left[\delta D + D\delta + (\bar{\rho} - 4\rho)\delta + 4\bar{\alpha}(D + \bar{\rho} - 2\rho)\right]\Phi_0. \end{aligned} \tag{13}$$

A third expression for a derivative of the spin coefficient κ is available from the NP equation (4.2b):

$$\delta\kappa = D\sigma - \left(\rho + \bar{\rho}\right)\sigma + \left(\bar{\alpha} + 3\beta\right)\kappa. \tag{14}$$

The integrability conditions of the derivatives $\Delta\kappa$, $D\kappa$ and $\delta\kappa$ will be explored in the next section.

4. Master equations

In this section, we continue to use the gauge (11) in which Eq. (10) takes the form

$$\begin{aligned} & [\delta\bar{\delta} - D\Delta + (2\gamma - \mu)D - 2\alpha\delta + (\bar{\rho} + 2\rho)\Delta \\ & + 2\mu\rho - 2\gamma\left(2\rho + \bar{\rho}\right) - 2\delta\alpha + \Psi_2 + 2\bar{\Phi}_1\Phi_1]\Phi_0 = 0. \end{aligned} \tag{15}$$

Inserting the unperturbed quantities in Eq. (15), we get the first wave equation:

$$\Box_1 \Phi_{\mathbf{0}} = 0, \tag{16}$$

where we introduce the operator

$$\begin{aligned}
\Box_s = &\, \boldsymbol{\Delta}^{-s}\frac{\partial}{\partial r}\boldsymbol{\Delta}^{s+1}\frac{\partial}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + s\left(1 - s\frac{\cos^2\vartheta}{\sin^2\vartheta}\right) \\
&+ \left[2a\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right) + \frac{1}{\sin^2\vartheta}\left(\frac{\partial}{\partial\varphi} + 2is\cos\vartheta\right)\right]\frac{\partial}{\partial\varphi} \qquad (17) \\
&+ a^2\sin^2\vartheta\frac{\partial^2}{\partial t^2} - 2\left[(r^2+a^2)\frac{\partial}{\partial r} + (s+2)r + ia\cos\vartheta\right]\frac{\partial}{\partial t}.
\end{aligned}$$

The spectral condition $\Box_s\Phi_{\mathbf{0}} = \chi\Phi_{\mathbf{0}}$ is separable in the coordinates r and ϑ on the basis of mode functions of the form $f\left(r,\vartheta\right)e^{i(m\varphi-\omega t)}$. The homogeneous wave equation (16) can be brought to the form

$$\left[\left(\nabla^a + s\Gamma^a\right)\left(\nabla_a + s\Gamma_a\right) - 4s^2\Psi_2\right]\Phi_0 = 0 \qquad (18)$$

with Ψ_2 is as given in Eq.(6), $s = 1$ and

$$\Gamma^r = \tfrac{2}{\zeta}, \qquad \Gamma^\vartheta = 0, \qquad \Gamma^\varphi = i\cos\vartheta + a\sin^2\vartheta\tfrac{\zeta\overline{\zeta}+\mathrm{m}(2\overline{\zeta}+\zeta)-2\mathrm{e}^2}{\zeta^2\overline{\zeta}}, \qquad \Gamma^t = \tfrac{\zeta\overline{\zeta}-\mathrm{m}(2\overline{\zeta}+\zeta)+2\mathrm{e}^2}{\zeta^2\overline{\zeta}}. \qquad (19)$$

Equations in this class have been studied by Bini, Cherubini, Jantzen and RuffiniBini (2002).

The second wave equation arises as follows. Given the three different derivatives (11), (13) and (14) of κ, we can derive three equations from the commutation relations. Only one of these equations is a second order differential equation decoupled from the other variables. It is obtained by applying the NP commutator

$$\delta\Delta - \Delta\delta = -\overline{\nu}D + (\tau - \bar{\alpha} - \beta)\,\Delta - \overline{\lambda}\bar{\delta} - (\mu - \gamma + \overline{\gamma})\,\delta \qquad (20)$$

to κ, and substituting the second derivatives $\delta\Delta\Phi_1$ and $\delta\bar{\delta}\Phi_1$ from the NP commutators (20) and

$$\bar{\delta}\delta - \delta\bar{\delta} = (\overline{\mu} - \mu)\,D + (\overline{\rho} - \rho)\,\Delta - (\bar{\alpha} - \beta)\,\bar{\delta} - \left(\overline{\beta} - \alpha\right)\delta \qquad (21)$$

for Φ_1. The resulting equation has the form

$$\mathcal{D}_1\sigma + \mathcal{D}_2\Phi_0 + \mathcal{A}\kappa = 0, \qquad (22)$$

where

$$\begin{aligned}
\mathcal{D}_1 &= 2\{\bar{\Phi}_2\Phi_1[\delta\bar{\delta} - \Delta D - 4\alpha\delta + 2\bar{\alpha}\bar{\delta} + (\bar{\rho}+\rho)\Delta \\
&\quad + \bar{\gamma}\rho + 4\bar{\Phi}_1\Phi_1 - 8\alpha\bar{\alpha} + \Delta\bar{\rho} - 4\delta\alpha + (\bar{\rho} - 2\rho)\bar{\mu} \\
&\quad + 3\,(\mu-\gamma)\,\rho - (4\gamma-\mu)\bar{\rho} + 5\Psi_2] \\
&\quad - \Phi_1\left[\bar{\mu}\bar{\Phi}_2 - 2\bar{\nu}\bar{\Phi}_1 - (4\gamma-\mu)\bar{\Phi}_2\right] D \\
&\quad + \bar{\Phi}_2\left[\bar{\delta}\Phi_1(\delta+4\bar{\alpha}) - \Delta\Phi_1\,(D-2\rho)\right]\} \\
\mathcal{D}_2 &= \bar{\nu}\left[\bar{\Phi}_1(D\delta+\delta D) - \bar{\Phi}_2 DD\right] + \left[(\bar{\rho}-4\rho)\bar{\nu} + 4\bar{\Phi}_2\Phi_1\right]\bar{\Phi}_1\delta \\
&\quad + 4\left[(\bar{\nu}\rho - \bar{\Phi}_2\Phi_1)\bar{\Phi}_2 + \bar{\alpha}\bar{\nu}\bar{\Phi}_1\right] D + 4\left[(\bar{\rho}-2\rho)\bar{\nu} + 2\bar{\Phi}_2\Phi_1\right]\bar{\alpha}\bar{\Phi}_1 \\
\mathcal{A} &= 2\bar{\Phi}_2\Phi_1\left[\delta\,(3\gamma+\bar{\gamma}-\bar{\mu}) + 2(\Delta+\mu+\bar{\gamma}-\gamma)\bar{\alpha} + \bar{\nu}\rho - 4\bar{\Phi}_2\Phi_1\right].
\end{aligned} \tag{23}$$

The operators $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{A}$ can be taken to have their form in the Kerr-Newman space-time since they act on first-order functions. The term $\mathcal{A}\kappa$ in Eq. (22) vanishes.

It proves advantageous to introduce in (22) the '*news potential*' ψ by

$$\psi = \frac{\sigma}{\zeta^2}. \tag{24}$$

We then have the wave equation of the form

$$\Box_2\psi = \frac{1}{\bar{\zeta}^2 \mathrm{e}} J\Phi_0, \tag{25}$$

where

$$\begin{aligned}
J &= \left(\mathrm{e}^2 - \mathrm{m}\zeta\right)\left(\frac{\partial}{\partial\vartheta} + ia\sin\vartheta\frac{\partial}{\partial t} + \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \frac{\cos\vartheta}{\sin\vartheta}\right)\frac{\partial}{\partial r} \\
&\quad - \frac{1}{\bar{\bar{\zeta}}}\left[\mathrm{e}^2 + \mathrm{m}\left(2\bar{\zeta}-\zeta\right)\right]\left(\frac{\partial}{\partial\vartheta} + ia\sin\vartheta\frac{\partial}{\partial t} + \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \frac{\cos\vartheta}{\sin\vartheta}\right) \\
&\quad + \frac{1}{\bar{\bar{\zeta}}}\left[\mathrm{e}^2 - \mathrm{m}\left(2\bar{\zeta}+\zeta\right)\right] ia\sin\vartheta\frac{\partial}{\partial r}.
\end{aligned} \tag{26}$$

Both equations (15) and (25) pick up an additional source term in the presence of other form of matter, and these terms contain the corresponding stress-energy tensor T_{ab}. Following Teukolsky (1973), we expand the homogeneous solutions in quasi-normal modes with energy ω and helicity m :

$$\Phi_0 = \int d\omega \sum_{l,m} R\,(r)\, S_l^m\,(\vartheta)\, e^{i(m\varphi-\omega t)}. \tag{27}$$

Separating the kernel of the operator $\Box_s$, the radial function $R\,(r)$ and the angular function $S_l^m\,(\vartheta)$ satisfy the ordinary differential equations, respectively,

$$\left[\mathbf{\Delta}^{-s}\frac{\partial}{\partial r}\mathbf{\Delta}^{s+1}\frac{\partial}{\partial r}+2i\left[(r^2+a^2)\omega-am\right]\frac{\partial}{\partial r}+2i\omega(s+2)r+2am\omega-\Lambda\right]R = 0 \tag{28}$$

$$\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}-\left(\frac{m+s\cos\vartheta}{\sin\vartheta}\right)^2+(\omega a\cos\vartheta-1)^2+s-1+\Lambda-\omega^2a^2\right]S = 0 \tag{29}$$

and Λ is the separation constant. Solution techniques for equations of the type (28) and (29) have been developed in Mano (1996, 1997).

5. Boundary conditions

At the horizon, $\mathbf{\Delta}=0$, the radial equation (28) is singular, even though the metric (3) remains here regular. We shall impose the boundary conditions at null infinity, $r\to\infty$ and on the horizon. To this end, we perform a transformation of both the dependent variable and the radial coordinate:

$$Z = e^{iA}\mathbf{\Delta}^{s/2}(r^2+a^2)^{1/2}R\ , \qquad \frac{\partial r'}{\partial r}=\frac{r^2+a^2}{\mathbf{\Delta}}, \tag{30}$$

where Z and r' are the new radial function and coordinate, respectively. We choose the real function A as follows,

$$\mathbf{\Delta}A_{,r}=(r^2+a^2)\omega-am. \tag{31}$$

As a result, the first derivatives of Z do not appear in the radial equation

$$Z_{,r'r'}+UZ=0 \tag{32}$$

with

$$U=\frac{Wr\,[3Wr+2\,(\mathrm{m}-r)]-V^2}{(r^2+a^2)^2} + \left[2(s+1)ir\omega+2am\omega-\Lambda-s-W\right]\frac{W}{r^2+a^2} \tag{33}$$

$$V=(r-\mathrm{m})s+i\left[(r^2+a^2)\omega-am\right] \tag{34}$$

$$W=\frac{\mathbf{\Delta}}{r^2+a^2}. \tag{35}$$

(i) In the neighborhood of null infinity, for large values of r, we expand the equation (28) in powers of $1/r$:

$$Z_{,r'r'} + \left(\omega^2 + \frac{2i\omega}{r}\right) Z \approx 0. \tag{36}$$

The asymptotic solutions of (36) and (31) are $Z \sim r^{\pm 1} e^{\mp i\omega r'}$ and the asymptotic form of A is $A \sim \omega r$. Hence the asymptotic form of the radial function is $R \sim r^{-s} e^{-2i\omega r'}$ and r^{-s-2}.

(ii) At the event horizon, $r = r_+$ or $r' \to -\infty$, the radial equation becomes

$$Z_{,r'r'} - \frac{V^2}{(a^2 + r_+^2)^2} Z \approx 0. \tag{37}$$

The boundary conditions on the event horizon will be selected by taking into account the absence or presence of incoming or outgoing radiation.

The boundary condition for the angular equation (29), imposed at $\vartheta = 0$ and at $\vartheta = \pi$ is that S must be regular on the boundary. As before (Teukolsky 1973), this leads to the Sturm-Liouville eigenvalue problem and yields a complete set of complex solutions.

6. Ramifications

The Maxwell equations for electromagnetic perturbations on a fixed space-time are homogeneous. The circularly polarized normal modes have the structure $\phi_A\,(r, \vartheta)\, e^{i(m\varphi - \omega t)}$. The interaction with the gravitational field, via the stress-energy tensor $\phi_A \overline{\phi}_B$, introduces a mixing among the electromagnetic modes, even in the linearized theory. Thus, the full electrovacuum perturbation equations cease to be homogeneous in the modes. In quantum field theory, perturbative methods are available for such interacting systems. Strangely, no comparable treatment for the corresponding classical system is known. This regrettable backlog in the classical theory, however, will help illuminate the origin of the long-standing difficulties in finding a separable equation for the perturbations.

In this paper, the following picture emerges for the classical electrovacuum perturbations. There exists a *subset* of the field equations, consisting, in part, of gravitational equations and of some of the Maxwell equations not containing any mode mixing. (This is because the complex conjugate electromagnetic stresses are absent from these equations). From these relations alone, it is possible to obtain a *pair* of wave equations for the quantities Φ_0 and σ. Thus a normal mode expansion for this doublet of fields *is* available.

Given the perturbed quantities Φ_0 and σ, Eq. (13) is an inhomogeneous first-order ordinary differential equation for the function κ. Integration of Eq. (13) yields the r dependence of the spin coefficient κ. Once this is known, the r dependence of expressions like $\Delta\kappa$ is

fixed. We then get Ψ_1 by solving the linear algebraic Eq. (11) for Ψ_1. Mode mixing occurs only in the further gravitational and electromagnetic perturbation components.

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